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# A new aspect of the cozero lattice in pointfree topology

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ABSTRACT

The cozero part $\operatorname{Coz} L$ of a frame $L$ has been studied quite extensively, but invariably from the point of view that it is a $\sigma$-frame – a fact proved with the aid of the Axiom of Countable Choice. Here, it will be shown that, for certain purposes, the latter is not required. For this a new description is presented of the *realcompletion* of a completely regular frame in terms of $\operatorname{Coz} L$ which does not involve any choice principle. The key to this is the introduction of a particular type of ideal in $\operatorname{Coz} L$ which amounts to a choice-free form of the $\sigma$-ideals usually considered in this context.



As a familiar tool in classical topology, the lattice of *zero-sets* of a topological space $X$ (meaning: the sets $Z(f) = \{x \in X \mid f(x) = 0\}$ for the continuous real-valued functions $f$ on $X$) plays an important rôle in connection with various aspects of Tychonoff spaces. On the other hand, in the context of pointfree topology, that is, for frames (or, alternatively, locales) it is natural that this rôle is taken over by the dual notion, the *cozero lattice* $\operatorname{Coz} L$ of a frame $L$, consisting of the elements $c \in L$ of the form

$$c = \operatorname{coz}(\gamma) = \gamma\big((-,0) \vee (0,-)\big) = \bigvee\big\{\gamma(p,0) \vee \gamma(0,q) \bigm| p < 0 < q \text{ in } \mathbf{Q}\big\}$$

for some real-valued continuous function $\gamma$ on $L$.

$\operatorname{Coz} L$ has been investigated quite extensively, but invariably from the point of view that it is a $\sigma$-frame or, more precisely, a sub-$\sigma$-frame of the ambient frame $L$ – a fact one proves by invoking ACC, the Axiom of Countable Choice. It is the aim of this note to show that, for certain crucial purposes, the latter is not required – somewhat contrary to existing perceptions. We do this by presenting, among other things, a new description of the *$\mathcal{R}$-complete coreflection* of a completely regular frame $L$ in terms of $\operatorname{Coz} L$ which does not involve ACC, valid in Zermelo–Fraenkel set theory.

To this end, we introduce the notion of *archimedean ideals* of the lattice $\operatorname{Coz} L$ as those lattice ideals $J$ such that $\operatorname{coz}(\alpha) \in J$ whenever $\alpha \geqslant \mathbf{0}$ and $\operatorname{coz}((n\alpha - \beta)^{+}) \in J$ for all $n$ with some suitable $\beta \geqslant \mathbf{0}$, and show that they constitute a frame which provides the coreflection in question (Proposition 1). In addition, we use this result to obtain new characterizations of the $\mathcal{R}$-complete and the realcompact frames (Propositions 2 and 3) which amount to choice-free forms of criteria given originally by Banaschewski–Gilmour [5] based on ACC. Further, we consider the "discrete" counterparts of these results, based on the notion of integer-valued continuous functions, here specifically considered for zero-dimensional frames. In this setting, the Boolean part $BL$ of the frame $L$ takes over the rôle of $\operatorname{Coz} L$, and we derive a new description of the $\mathcal{Z}$-complete $L$ in terms of $BL$ (Proposition 4). Finally, we link the results concerning $\mathcal{R}$-completeness with the familiar ones based on ACC by showing among other things that, given the latter, the $\sigma$-ideals are the same as the archimedean ideals (Proposition 5).

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## 1. Background

For general facts concerning frames we refer to Johnstone [7], Pultr [9], or Vickers [10]. Details regarding the ring of continuous real-valued functions on a frame can be found in Banaschewski [2,4]. Here, we restrict ourselves to a brief outline of the facts specifically needed in the present context.

Regarding the function rings involved, the starting point is the *frame* $\mathcal{L}(\mathbf{R})$ *of reals*, defined by generators and relations, the former being all pairs $(p,q)$ of rationals while the latter are as follows:

(R1) $(p,q) \wedge (r,s) = (p \vee r, q \wedge s)$,

(R2) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leqslant r < q \leqslant s$,

(R3) $(p,q) = \bigvee \{(r,s) \mid p < r < s < q\}$,

(R4) $e = \bigvee \{(p,q) \mid \text{all } p,q \in \mathbf{Q}\}$

where $e$, as always, is the unit (= top).

Now, the continuous real-valued functions on a frame $L$ are the homomorphisms $\alpha, \beta, \ldots : \mathcal{L}(\mathbf{R}) \to L$ which come equipped with the following algebraic operations, derived from those of $\mathbf{Q}$ as lattice-ordered ring:

For $\diamond = +, \cdot, \wedge, \vee$:

$$\alpha \diamond \beta(p,q) = \bigvee \{\alpha(r,s) \wedge \beta(t,u) \mid \langle r,s\rangle \diamond \langle t,u\rangle \subseteq \langle p,q\rangle\}$$

where $\langle \cdot,\cdot \rangle$ stands for the open interval in $\mathbf{Q}$ and the given condition means that $x \diamond y \in \langle p,q\rangle$ for any $x \in \langle r,s\rangle$ and $y \in \langle t,u\rangle$.

$$(-\alpha)(p,q) = \alpha(-q,-p).$$

For each $r \in \mathbf{Q}$, the corresponding constant function $\mathbf{r}$ such that

$$\mathbf{r}(p,q) = e \quad \text{if } p < r < q \quad \text{and} \quad \mathbf{r}(p,q) = 0 \quad \text{otherwise}.$$

$\mathcal{R}L$ will then be the resulting algebraic system which is easily seen to satisfy all identities valid for the corresponding operations in $\mathbf{Q}$ making it a commutative $f$-ring with unit. Moreover, it is archimedean.

Further, the correspondence $L \mapsto \mathcal{R}L$ is functorial, the $\ell$-ring homomorphism $\mathcal{R}h : \mathcal{R}L \to \mathcal{R}M$ induced by a frame homomorphism $h : L \to M$ being the map $\alpha \mapsto h\alpha$. Our specific concern here will be the resulting functor $\mathcal{R} : \mathbf{CRFrm} \to \mathbf{A}$ from the category of completely regular frames into that of archimedean $f$-rings with unit. Accordingly, all frames considered are taken to be completely regular, a natural restriction familiar from classical topology.

Connected with $\mathcal{R}$ we then have the following notions in **CRFrm**:

(1) a homomorphism $h : M \to L$ is called an $\mathcal{R}$*-isomorphism* if $\mathcal{R}h$ is an isomorphism,
(2) $L$ is called $\mathcal{R}$*-complete* if any $\mathcal{R}$-isomorphism $M \to L$ is actually an isomorphism, and
(3) an $\mathcal{R}$*-completion* of $L$ is an $\mathcal{R}$-isomorphism $M \to L$ with $\mathcal{R}$-complete $M$.

The basic result in this context is that any $L \in \mathbf{CRFrm}$ has an $\mathcal{R}$-completion, unique up to a unique isomorphism, which is also the coreflection map to $L$ from $\mathcal{R}$-complete frames.

To put these notions in perspective, their relation to the following concepts in classical topology should be pointed out: (1) corresponds to the dense $C$-embedding of Tychonoff spaces, (2) expresses an aspect of realcompactness, and hence (3) is the counterpart of Hewitt's realcompactification $\upsilon X$.

The procedure of obtaining the $\mathcal{R}$-completion most closely related to the usual description of the corresponding classical entity is as follows. The fundamental tool is the cozero map $\mathrm{coz} : \mathcal{R}L \to L$, defined by

$$\mathrm{coz}(\alpha) = \alpha\big((-,0) \vee (0,-)\big)$$

for $(-,0) = \bigvee\{(p,0) \mid p < 0 \text{ in } \mathbf{Q}\}$ in $\mathcal{L}(\mathbf{R})$ and the analogous $(0,-)$, which satisfies the following basic rules:

$$\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta), \qquad \mathrm{coz}(\mathbf{1}) = e,$$

$$\mathrm{coz}(\alpha+\beta) \leqslant \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta), \qquad \mathrm{coz}(\mathbf{0}) = 0,$$

$$\mathrm{coz}(\alpha) \leqslant \mathrm{coz}(\beta) \quad \text{if } \mathbf{0} \leqslant \alpha \leqslant \beta.$$

From these, one derives several further consequences such as

$$\mathrm{coz}(\alpha) = \mathrm{coz}\big(|\alpha|\big),$$

$$\mathrm{coz}(\alpha+\beta) = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta) = \mathrm{coz}(\alpha \vee \beta) \quad \text{if } \alpha, \beta \geqslant \mathbf{0}.$$

Now, the *cozero part* $\operatorname{Coz} L = \{\operatorname{coz}(\alpha) \mid \alpha \in \mathcal{R}L\}$ of $L$ is immediately seen to be a bounded sublattice of $L$ by the above rules. Furthermore, *if ACC is assumed*, one proves that $\operatorname{Coz} L$ is a sub-$\sigma$-frame of $L$. As a result one has the *frame* $\mathfrak{H}\operatorname{Coz} L$ *of its $\sigma$-ideals* and can then show that taking joins in $L$ provides a homomorphism $\mathfrak{H}\operatorname{Coz} L \to L$ which is the $\mathcal{R}$-completion of $L$.

As an added feature, one obtains that this is also the coreflection map from the completely regular Lindelöf frames – using ACC again, which in fact is equivalent to the condition that Lindelöf $= \mathcal{R}$-complete (Banaschewski [3, Proposition 9]).

It should be emphasized that there are other ways to obtain the $\mathcal{R}$-completion which are choice-free (Banaschewski [4]) but so far none are known that use a quotient frame of the frame $\mathfrak{J}\operatorname{Coz} L$ of ideals of $\operatorname{Coz} L$ in the style of $\mathfrak{H}\operatorname{Coz} L$ when ACC is given. It is the purpose of this paper to describe a way which does exactly that.

There is a further result about the cozero map which is needed for this, as follows.

**Lemma 1.** *For any $\alpha, \beta \geqslant \mathbf{0}$ in $\mathcal{R}L$,*

$$\operatorname{coz}(\alpha) = \bigvee \big\{\operatorname{coz}\big((n\alpha - \beta)^+\big) \bigm| n = 1, 2, \ldots\big\}.$$

**Proof.** First, some general observations: for any $\gamma \in \mathcal{R}L$

(i) $\operatorname{coz}(\gamma^+) = \gamma^+(0, -) = \gamma(0, -)$, and
(ii) $n\gamma(p, q) = \gamma(\frac{p}{n}, \frac{q}{n})$ for $n = 1, 2, \ldots$.

To obtain (i), use the definition $\gamma^+ = \gamma \vee \mathbf{0}$ and then calculate on the basis of the formula for $\alpha \vee \beta$. Similarly, calculate the product $\mathbf{n}\gamma$, using the definition of constant functions.

Now, put $c = \bigvee\{\operatorname{coz}((n\alpha - \beta)^+ \mid n = 1, 2, \ldots\}$. To see that $c \leqslant \operatorname{coz}(\alpha)$ note that $(n\alpha - \beta)^+ \leqslant n\alpha$ trivially and hence by the rules for coz

$$\operatorname{coz}\big((n\alpha - \beta)^+\big) \leqslant \operatorname{coz}(n\alpha) = \operatorname{coz}(\alpha).$$

For the reverse relation, note that

$$\operatorname{coz}\big((n\alpha - \beta)^+\big) = (n\alpha - \beta)(0, -) = \left(\alpha + \frac{-1}{n}\beta\right)(0, -) = \bigvee\left\{\alpha(r, s) \wedge \frac{-1}{n}\beta(t, u) \;\middle|\; \langle r, s\rangle + \langle t, u\rangle \subseteq \langle 0, -\rangle\right\},$$

the first two steps by our original observation, while the condition for the join amounts to $0 \leqslant r + t$. Hence any $r > 0$ and $t = -r$ are possible here so that

$$\operatorname{coz}\big((n\alpha - \beta)^+\big) \geqslant \bigvee\left\{\alpha(r, s) \wedge \frac{-1}{n}\beta(-r, u) \;\middle|\; r \leqslant s, -r \leqslant u\right\} = \alpha(r, -) \wedge \frac{-1}{n}\beta(-r, -) = \alpha(r, -) \wedge \beta(-, nr),$$

and taking join over all $n$ then shows

$$c \geqslant \alpha(r, -)$$

since $\bigvee\{\beta(-, nr) \mid n = 1, 2, \ldots\} = e$ by (R4) because $r > 0$. Finally, taking join over all $r > 0$ proves $c \geqslant \alpha(0, -) = \operatorname{coz}(\alpha)$. $\square$

**Remark 1.** The above relation already appears in Banaschewski [4], stated to result by a standard calculation, but on second thought it seemed advisable to be more explicit here and provide the details.

## 2. Archimedean ideals in Coz *L*

For any completely regular frame $L$, an ideal $J$ in the lattice $\operatorname{Coz} L$ will be called *archimedean* provided that $\operatorname{coz}((n\alpha - \beta)^+) \in J$ for some $\alpha, \beta \geqslant \mathbf{0}$ in $\mathcal{R}L$ and all $n$ implies $\operatorname{coz}(\alpha) \in J$.

To explain this terminology, recall that an *archimedean kernel* of an $\ell$-ring $A$ is an $\ell$-ideal $K$ in $A$ such that $A/K$ is archimedean, where the latter obviously holds iff $(na - b)^+ \in K$ for some $a, b \geqslant 0$ in $A$ and all $n$ implies $a \in K$. Now, for any ideal $J$ in $\operatorname{Coz} L$,

$$\operatorname{coz}^{-1}[J] = \big\{\gamma \in \mathcal{R}L \bigm| \operatorname{coz}(\gamma) \in J\big\}$$

is an $\ell$-ideal of $\mathcal{R}L$, and *this is an archimedean kernel iff $J$ is an archimedean ideal*. To see this, assume that $\tilde{J} = \operatorname{coz}^{-1}[J]$ is an archimedean kernel and let $\operatorname{coz}((n\alpha - \beta)^+) \in J$ for some $\alpha, \beta \geqslant 0$ and all $n$. Then $(n\alpha - \beta)^+ \in \tilde{J}$, hence $\alpha \in \tilde{J}$ so that $\operatorname{coz}(\alpha) \in J$, showing $J$ is archimedean. Conversely, given the latter and $(n\alpha - \beta)^+ \in \tilde{J}$ for some $\alpha, \beta \geqslant 0$ and all $n$ then $\operatorname{coz}((n\alpha - \beta)^+) \in J$, hence $\operatorname{coz}(\alpha) \in J$ and consequently $\alpha \in \tilde{J}$, making $\tilde{J}$ an archimedean kernel.

The archimedean ideals of $\operatorname{Coz} L$ clearly form a closure system, to be denoted $\mathfrak{A}\operatorname{Coz} L$. Note that $\{\operatorname{coz}(\gamma) \mid \operatorname{coz}(\gamma) \leqslant a\}$ is trivially an archimedean ideal for any $a \in L$ by Lemma 1; in particular, the principal ideals $\downarrow\operatorname{coz}(\alpha)$ of $\operatorname{Coz} L$ are archimedean. On the other hand, examples of ideals in $\operatorname{Coz} L$ which are not archimedean are easy to find. For this, consider the ideal

$J = \bigcup\{\downarrow\mathrm{coz}((\gamma - \frac{1}{n})^+) \mid n = 1, 2, \ldots\}$, for any $\gamma \geqslant \mathbf{0}$, which is archimedean iff $\mathrm{coz}(\gamma) = \mathrm{coz}((\gamma - \frac{1}{n})^+)$ for some $n$. Now, if this holds for $L = \mathfrak{O}X$, the frame of open sets of some space $X$, and $u \in C(X)$ corresponding to $\gamma$, that is

$$\gamma(p, q) = u^{-1}\big[\{\lambda \in \mathbf{R} \mid p < \lambda < q\}\big],$$

then $u(x) > 0$ implies $u(x) > \frac{1}{n}$ for some $n$, a condition which does not hold for $X = \mathbf{R}$ and $u(x) = \frac{1}{1+x^2}$.

The basic facts concerning $\mathfrak{A}\,\mathrm{Coz}\,L$ are now summed up as follows.

**Lemma 2.**

(i) *$\mathfrak{A}\,\mathrm{Coz}\,L$ is a completely regular frame and $L \mapsto \mathfrak{A}\,\mathrm{Coz}\,L$ is functorial.*
(ii) *Taking joins in $L$ provides a frame homomorphism $j_L : \mathfrak{A}\,\mathrm{Coz}\,L \to L$ which is an $\mathcal{R}$-isomorphism, natural in $L$.*
(iii) *For any $\mathcal{R}$-isomorphism $h : L \to M$ between completely regular frames the corresponding $\mathfrak{A}\,\mathrm{Coz}\,L \to \mathfrak{A}\,\mathrm{Coz}\,M$ is an isomorphism.*

**Proof.** (i) To see that $\mathfrak{A}\,\mathrm{Coz}\,L$ is a frame, consider the operator $k_0$ on the frame $\mathfrak{J}\,\mathrm{Coz}\,L$ of *all* ideals of $\mathrm{Coz}\,L$ such that

$$k_0(J) = \big\{\mathrm{coz}(\alpha) \bigm| \alpha \geqslant \mathbf{0}, \mathrm{coz}\big((n\alpha - \beta)^+\big) \in J \text{ for some } \beta \geqslant \mathbf{0}, \text{ all } n\big\}.$$

Then obviously $\mathfrak{A}\,\mathrm{Coz}\,L = \mathrm{Fix}(k_0)$. We claim $k_0$ is a *prenucleus* on $\mathfrak{J}\,\mathrm{Coz}\,L$, meaning:

$$k_0(J) \text{ is an ideal} \quad \text{and} \quad J \subseteq k_0(J),$$

$$I \subseteq J \quad \text{imples} \quad k_0(I) \subseteq k_0(J), \qquad k_0(I) \cap J \subseteq k_0(I \cap J)$$

for any $I, J \in \mathfrak{J}\,\mathrm{Coz}\,L$. This will then prove that the closure operator $k$ associated with $\mathfrak{A}\,\mathrm{Coz}\,L$ is a nucleus, making the latter a frame and $k$ a frame homomorphism $\mathfrak{J}\,\mathrm{Coz}\,L \to \mathfrak{A}\,\mathrm{Coz}\,L$ (Banaschewski [1]).

First, $k_0(J)$ is an ideal. For any $\mathrm{coz}(\alpha)$ and $\mathrm{coz}(\gamma)$ in $k_0(J)$ where $\alpha, \gamma \geqslant \mathbf{0}$, we have $\mathrm{coz}((n\alpha - \beta)^+)$, $\mathrm{coz}((n\gamma - \delta)^+) \in J$ for all $n$ with some $\beta, \delta \geqslant \mathbf{0}$ by definition. Consequently, since

$$n(\alpha \vee \gamma) - (\beta \vee \delta) = \big(n\alpha - (\beta \vee \delta)\big) \vee \big(n\gamma - (\beta \vee \delta)\big) \leqslant (n\alpha - \beta) \vee (n\gamma - \delta)$$

it follows that

$$\mathrm{coz}\big(\big(n(\alpha \vee \gamma) - (\beta \vee \delta)\big)^+\big) \leqslant \mathrm{coz}\big((n\alpha - \beta)^+\big) \vee \mathrm{coz}\big((n\gamma - \delta)^+\big) \in J,$$

and hence $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha \vee \gamma) \in k_0(J)$. Similarly, if $\mathrm{coz}(\alpha) \in k_0(J)$ so that $\alpha \geqslant \mathbf{0}$ and $\mathrm{coz}((n\alpha - \beta)^+) \in J$ for all $n$ with some $\beta \geqslant \mathbf{0}$ then

$$\mathrm{coz}\big((n\alpha\gamma - \beta\gamma)^+\big) = \mathrm{coz}\big((n\alpha - \beta)^+\big) \wedge \mathrm{coz}(\gamma) \in J$$

for any $\gamma \geqslant \mathbf{0}$ in $\mathcal{R}L$, showing that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\gamma) = \mathrm{coz}(\alpha\gamma) \in k_0(J)$. In all, this makes $k_0(J)$ an ideal.

Further, regarding the three conditions listed for $k_0$, if $\mathrm{coz}(\alpha) \in J$ for some $\alpha \geqslant \mathbf{0}$ then $\mathrm{coz}((n\alpha - \beta)^+) \in J$ for any $n$ and $\beta \geqslant \mathbf{0}$ as already noted, and this immediately implies the first and the third of these while the second one is obvious.

Next, $\mathfrak{A}\,\mathrm{Coz}\,L$ is completely regular. To see this, note first that

$$\downarrow\mathrm{coz}(\alpha) = \bigvee\big\{\downarrow\mathrm{coz}\big((\alpha - \mathbf{r})^+\big) \bigm| 0 < r \in \mathbf{Q}\big\}$$

in $\mathfrak{A}\,\mathrm{Coz}\,L$ for any $\alpha \geqslant \mathbf{0}$: if an archimedean ideal $J$ contains all $\mathrm{coz}((\alpha - \mathbf{r})^+)$ then it contains all $\mathrm{coz}((n\alpha - \mathbf{1})^+ = \mathrm{coz}((\alpha - \frac{\mathbf{1}}{n})^+)$ and hence $\mathrm{coz}(\alpha)$. Now, for any $r > 0$ in $\mathbf{Q}$,

$$\mathrm{coz}\big((\alpha - \mathbf{r})^+\big) \wedge \mathrm{coz}\big((\mathbf{r} - \alpha)^+\big) = \mathrm{coz}(\mathbf{0}) = 0$$

while

$$\mathrm{coz}(\alpha) \vee \mathrm{coz}\big((\mathbf{r} - \alpha)^+\big) = \mathrm{coz}\big(\alpha + (\mathbf{r} - \alpha)^+\big) = \mathrm{coz}(\mathbf{r} \vee \alpha) = e$$

showing that $\downarrow\mathrm{coz}((\alpha - \mathbf{r})^+) \prec \downarrow\mathrm{coz}(\alpha)$ in $\mathfrak{A}\,\mathrm{Coz}\,L$ but then the same holds for the relation $\prec\!\!\prec$ by the properties of $\mathbf{Q}$, and since the $\downarrow\mathrm{coz}(\alpha)$ obviously generate $\mathfrak{A}\,\mathrm{Coz}\,L$ this proves the claim.

Finally, the correspondence $L \mapsto \mathfrak{A}\,\mathrm{Coz}\,L$ is functorial, as a consequence of the familiar fact that $L \mapsto \mathfrak{J}\,\mathrm{Coz}\,L$ is functorial together with the specific nature of the nucleus which determines $\mathfrak{A}\,\mathrm{Coz}\,L$. Thus, for $h : M \to L$, the corresponding $\bar{h} : \mathfrak{J}\,\mathrm{Coz}\,M \to \mathfrak{J}\,\mathrm{Coz}\,L$ is

$$J \mapsto \bigcup\big\{\downarrow h\big(\mathrm{coz}(\alpha)\big) \bigm| \mathrm{coz}(\alpha) \in J\big\},$$

and if $k_0^M$ and $k_0^L$ are the relevant prenuclei then $\bar{h}k_0^M(J) \subseteq k_0^L\bar{h}(J)$ because $\mathrm{coz}((n\alpha-\beta)^+) \in J$ implies $\mathrm{coz}((n(h\alpha)-(h(\beta))^+ \in \bar{h}(J)$ so that $h(\mathrm{coz}(\alpha)) = \mathrm{coz}(h\alpha) \in k_0^L\bar{h}(J)$. Further, it follows by general principles that also $\bar{h}k^M \leqslant k^L\bar{h}$ for the corresponding nuclei, and hence we have the commuting square

$$\begin{tikzcd}
\mathfrak{J}\,\mathrm{Coz}\,M \arrow[r, "\bar{h}"] \arrow[d, "k^M"'] & \mathfrak{J}\,\mathrm{Coz}\,L \arrow[d, "k^L"] \\
\mathfrak{A}\,\mathrm{Coz}\,M \arrow[r, "\tilde{h}"'] & \mathfrak{A}\,\mathrm{Coz}\,L
\end{tikzcd}$$

where $\tilde{h} : J \mapsto k^L\bar{h}(J)$ is the desired frame homomorphism. That the correspondence $h \mapsto \tilde{h}$ then has the required properties results from the analogous fact for $h \mapsto \bar{h}$ and the expression of $\tilde{h}$ in terms of $\bar{h}$.

(ii) Since the maps $\mathfrak{J}\,\mathrm{Coz}\,L \to L$ by taking joins in $L$ are frame homomorphisms, the same follows for the $j_L : \mathfrak{A}\,\mathrm{Coz}\,L \to L$ induced by them since $\bigvee k_0^L(J) = \bigvee J$ for any $J \in \mathfrak{J}\,\mathrm{Coz}\,L$: if $\mathrm{coz}(\alpha) \in k_0^L(J)$ for some $\alpha \geqslant \mathbf{0}$ so that $\mathrm{coz}((n\alpha-\beta)^+) \in J$ for some $\beta \geqslant \mathbf{0}$ and all $n$ then $\mathrm{coz}(\alpha) \leqslant \bigvee J$ by Lemma 1. Further, the naturality of the $j_L$ is a consequence of that of the maps $\mathfrak{J}\,\mathrm{Coz}\,L \to L$ in a similar way: for any $h : M \to L$ we have the diagram

$$\begin{tikzcd}
\mathfrak{J}\,\mathrm{Coz}\,M \arrow[r, "\bar{h}"] \arrow[d, "k^M"'] & \mathfrak{J}\,\mathrm{Coz}\,L \arrow[d, "k^L"] \\
\mathfrak{A}\,\mathrm{Coz}\,M \arrow[r, "\tilde{h}"] \arrow[d, "j_M"'] & \mathfrak{A}\,\mathrm{Coz}\,L \arrow[d, "j_L"] \\
M \arrow[r, "h"'] & L
\end{tikzcd}$$

(using previous notation) in which both the top and the outer square commute which makes the bottom square commute, as claimed.

Finally, to see that $j_L$ is an $\mathcal{R}$-isomorphism, note first that any $\alpha : \mathcal{L}(\mathbf{R}) \to L$ actually maps the generators $(p,q)$ of $\mathcal{L}(\mathbf{R})$ into $\mathrm{Coz}\,L$ by the familiar rule

$$\alpha(p,q) = \mathrm{coz}\big((\alpha-\mathbf{p})^+ \wedge (\mathbf{q}-\alpha)^+\big)$$

(Banaschewski [2]) and consequently determines a map

$$(p,q) \mapsto \beta(p,q) = {\downarrow}\alpha(p,q)$$

into $\mathfrak{A}\,\mathrm{Coz}\,L$. We claim this defines a homomorphism $\beta : \mathcal{L}(\mathbf{R}) \to \mathfrak{A}\,\mathrm{Coz}\,L$. Regarding the defining relations (R1) and (R2) of $\mathcal{L}(\mathbf{R})$ there is nothing to prove since $\downarrow : \mathrm{Coz}\,L \to \mathfrak{A}\,\mathrm{Coz}\,L$ is a lattice homomorphism. Next, for the required identity

$${\downarrow}\alpha(p,q) = \bigvee\big\{{\downarrow}\alpha(r,s) \bigm| p<r<s<q\big\}$$

corresponding to (R3), note first that

$$\begin{aligned}
{\downarrow}\alpha(p,q) &= {\downarrow}\mathrm{coz}\big((\alpha-\mathbf{p})^+ \wedge (\mathbf{q}-\alpha)^+\big) \\
&= \bigvee\big\{{\downarrow}\mathrm{coz}\big(\big(n\big((\alpha-\mathbf{p})^+ \wedge (\mathbf{q}-\alpha)^+\big)-\mathbf{1}\big)^+\big) \bigm| n=1,2,\ldots\big\} \\
&= \bigvee\left\{{\downarrow}\mathrm{coz}\left(\left(\big((\alpha-\mathbf{p})^+ \wedge (\mathbf{q}-\alpha)^+\big)-\frac{\mathbf{1}}{n}\right)^+\right) \Bigm| n=1,2,\ldots\right\}.
\end{aligned}$$

Further by straightforward calculation

$$\left(\big((\alpha-\mathbf{p})^+ \wedge (\mathbf{q}-\alpha)^+\big)-\frac{1}{n}\right)^+ = \left(\alpha-\mathbf{p}-\frac{1}{n}\right)^+ \wedge \left(\mathbf{q}-\frac{1}{n}-\alpha\right)^+,$$

and the cozero element of this is $\alpha(p+\frac{1}{n}, q-\frac{1}{n})$. It follows that

$${\downarrow}\alpha(p,q) = \bigvee\left\{{\downarrow}\alpha\left(p+\frac{1}{n}, q-\frac{1}{n}\right) \Bigm| n=1,2,\ldots\right\},$$

which amounts to the desired identity.

Finally, given that

$$\alpha(-n,n) = \mathrm{coz}\big((\alpha+\mathbf{n})^+ \wedge (\mathbf{n}-\alpha)^+\big) = \mathrm{coz}\big(\big(\mathbf{n}-|\alpha|\big)^+\big)$$

for any $\alpha \in \mathcal{R}L$ and natural $n$, we find

$$\bigvee\big\{{\downarrow}\alpha(-n,n) \bigm| n=1,2,\ldots\big\} = \bigvee\big\{{\downarrow}\mathrm{coz}\big(\big(\mathbf{n}-|\alpha|\big)^+\big) \bigm| n=1,2,\ldots\big\} = {\downarrow}\mathrm{coz}(\mathbf{1}) = e$$

which settles the case of (R4). In all then we have $\beta \in \mathcal{R}(\mathfrak{A}\operatorname{Coz} L)$ as claimed, and since $j_L\beta = \alpha$ trivially this shows $j_L$ is an $\mathcal{R}$-surjection. On the other hand, $j_L$ is obviously dense which makes $\mathcal{R}j_L$ one–one and hence an isomorphism.

(iii) First, any $\mathcal{R}$-isomorphism $h : M \to L$ induces an isomorphism $\operatorname{Coz} M \to \operatorname{Coz} L$: it maps $\operatorname{Coz} M$ into $\operatorname{Coz} L$ (as any frame homomorphism does since $h(\operatorname{coz}(\alpha)) = \operatorname{coz}(h\alpha)$), and onto because any $\gamma \in \mathcal{R}L$ is an $h\alpha$, $\alpha \in \mathcal{R}M$, so that it remains to show the induced map is one–one.

Now, $h(\operatorname{coz}(\alpha)) = e$ implies $\operatorname{coz}(\mathcal{R}h(\alpha)) = e$, hence $\mathcal{R}h(\alpha)$ is *invertible* but then $\alpha$ is also invertible, $\mathcal{R}h$ being an isomorphism, and hence $\operatorname{coz}(\alpha) = e$. Next, if $h(\operatorname{coz}(\alpha)) = h(\operatorname{coz}(\beta))$ for some $\alpha, \beta \geqslant \mathbf{0}$ in $\mathcal{R}M$ then

$$\operatorname{coz}(\alpha) \vee \operatorname{coz}\big((r-\alpha)^+\big) = \operatorname{coz}\big(\alpha + (r-\alpha)^+\big) = \operatorname{coz}(r \vee \alpha) = e$$

for any $r > 0$ in $\mathbf{Q}$ and consequently

$$e = h\big(\operatorname{coz}(\alpha) \vee \operatorname{coz}\big((r-\alpha)^+\big)\big) = h\big(\operatorname{coz}(\beta) \vee \operatorname{coz}\big((r-\alpha)^+\big)\big)$$

so that $e = \operatorname{coz}(\beta) \vee \operatorname{coz}((r-\alpha)^+)$, as noted. It follows that $\operatorname{coz}((\alpha - r)^+) \leqslant \operatorname{coz}(\beta)$ and taking join over all $r > 0$ then shows $\operatorname{coz}(\alpha) \leqslant \operatorname{coz}(\beta)$, proving equality by symmetry.

From here, it is perfectly obvious that $J$ is archimedean iff $h[J]$ is archimedean, for any ideal $J$ of $\operatorname{Coz}(M)$, and this clearly implies that the homomorphism $\tilde{h} : \mathfrak{A}\operatorname{Coz} M \to \mathfrak{A}\operatorname{Coz} L$ determined by $h$ is an isomorphism. □

## 3. The $\mathcal{R}$-complete coreflection

Since the result involved here is a purely formal consequence of the properties of $\mathfrak{A}\operatorname{Coz} L$ given in Lemma 2 we first derive its general, abstract version.

For any class $\mathcal{A}$ of maps in a category $\mathbf{C}$, call $L \in \mathbf{C}$ *$\mathcal{A}$-complete* if any $M \to L$ in $\mathcal{A}$ is an isomorphism.

Now, let $\mathcal{A}$ be a class of monomorphisms in $\mathbf{C}$ and $S$ an endofunctor of $\mathbf{C}$ together with a natural transformation $\sigma : S \to Id_{\mathbf{C}}$ such that

(1) $Sh$ is an isomorphism for any $h \in \mathcal{A}$, and
(2) each $\sigma_L \in \mathcal{A}$.

Then we have

**Lemma 3.** *For any $L \in \mathbf{C}$, $\sigma_L : SL \to L$ is the coreflection map to $L$ from $\mathcal{A}$-complete objects as well as the only map $M \to L$ in $\mathcal{A}$, up to isomorphism, with $\mathcal{A}$-complete $M$.*

**Proof.** First, any $SL$ is $\mathcal{A}$-complete: for any $h : M \to SL$ in $\mathcal{A}$, $k = S(\sigma_L h) = (S\sigma_L)(Sh)$ is an isomorphism by (1) and (2); on the other hand, $\sigma_L k = \sigma_L h \sigma_M$ by naturality, hence $k = h\sigma_M$ by (2), and since $h$ is monic this makes it an isomorphism.

Next, for any $h : M \to L$ with $\mathcal{A}$-complete $M$, $h\sigma_M = \sigma_L Sh$ by naturality where $\sigma_M$ is an isomorphism by (2), hence $h = \sigma_L(Sh)\sigma_M^{-1}$ and this factorization by $\sigma_L$ is unique, again by (2).

Finally, if the given $h$ belongs to $\mathcal{A}$, then $Sh$ is also an isomorphism by (1) and $h = \sigma_L(Sh)\sigma_M^{-1}$ shows that $\sigma_L$ is unique up to isomorphism. □

Turning now to $\mathfrak{A}\operatorname{Coz} L$, note that any $\mathcal{R}$-isomorphism in **CRFrm** is monic because it is *dense*: for any $h : M \to L$ such that $\mathcal{R}h$ is an isomorphism, let $h(a) = 0$ and consider any $\operatorname{coz}(\alpha) \leqslant a$. Then $0 = h(\operatorname{coz}(\alpha)) = \operatorname{coz}(h\alpha) = \operatorname{coz}(\mathcal{R}h(\alpha))$, hence $\mathcal{R}h(\alpha) = \mathbf{0}$, but then also $\alpha = \mathbf{0}$ so that $\operatorname{coz}(\alpha) = 0$, and since $a$ is a join of cozero elements this shows $a = 0$.

It follows that the functor $\mathfrak{A}\operatorname{Coz} : \mathbf{CRFrm} \to \mathbf{CRFrm}$ and the class of $\mathcal{R}$-isomorphism satisfy the above conditions for $S$ and $\mathcal{A}$ so that Lemma 3 applies and we obtain

**Proposition 1.** *For any completely regular frame $L$, $j_L : \mathfrak{A}\operatorname{Coz} L \to L$ is the coreflection map from $\mathcal{R}$-complete frames as well as the only $\mathcal{R}$-isomorphism to $L$, up to isomorphism, with $\mathcal{R}$-complete domain.*

It may be of interest to compare this result with the alternative description of the $\mathcal{R}$-complete coreflection in terms of the $\ell$-ideals of $\mathcal{R}L$ presented in Banaschewski [4] as follows.

Surprisingly simple formal arguments show that the functor $\mathcal{R} : \mathbf{CRFrm} \to \mathbf{A}$ has a left adjoint $\mathfrak{S} : \mathbf{A} \to \mathbf{CRFrm}$, and the corresponding adjunction maps $\mathfrak{S}\mathcal{R}L \to L$ are then easily identified as the coreflection maps in question, Further, with substantially more effort, $\mathfrak{S}$ is described concretely as the functor $\mathfrak{K}$ associating with each archimedean $f$-ring $A$ with unit the frame $\mathfrak{K}A$ of its archimedean kernels while the adjunction maps $\mathfrak{K}\mathcal{R}L \to L$ take any archimedean kernel $K$ of $\mathcal{R}L$ to $\bigvee\{\operatorname{coz}(\alpha) \mid \alpha \in K\}$. Now, it follows from Proposition 1 that there are isomorphisms $\mathfrak{A}\operatorname{Coz} L \to \mathfrak{K}\mathcal{R}L$, compatible with the respective coreflection maps, and one can then show that they are actually given by the correspondence $J \mapsto \operatorname{coz}^{-1}[J]$ considered earlier.

On the other hand, these maps are readily seen to be isomorphisms without falling back on Proposition 1 which then makes the latter a consequence of the results about the functor $\mathcal{R}$ referred to above. Viewed in this way, the arguments given here amount to a selfcontained proof of the properties of $\mathfrak{A}\,\mathrm{Coz}\,L$ in question, circumventing the rather elaborate verification of the relevant properties of the functor $\mathfrak{K}$ referred to above.

For the record, we include the proof that the map $J \mapsto \mathrm{coz}^{-1}[J]$ defines an isomorphism $\mathfrak{A}\,\mathrm{Coz}\,L \to \mathfrak{K}\mathcal{R}L$. From earlier considerations, we already know that it maps $\mathfrak{A}\,\mathrm{Coz}\,L$ into $\mathfrak{K}\mathcal{R}L$. Further, it is obviously one–one: trivially, $J = \{\mathrm{coz}(\alpha) \mid \alpha \in \mathrm{coz}^{-1}[J]\}$ for any archimedean ideal $J$ of $\mathrm{Coz}\,L$, and preserves and reflects inclusion. Hence, it only remains to prove that it is *onto*. For this, let $K \subseteq \mathcal{R}L$ be any archimedean kernel and put

$$J = \mathrm{coz}[K] = \bigl\{\mathrm{coz}(\gamma) \bigm| \gamma \in K\bigr\}.$$

This is certainly an ideal of $\mathrm{Coz}\,L$: if $\gamma, \delta \in K$ then also $|\gamma| + |\delta| \in K$ and $\mathrm{coz}(|\gamma| + |\delta|) = \mathrm{coz}(\gamma) \vee \mathrm{coz}(\delta)$, and the second ideal property is seen similarly. Naturally, what one wants here is that $J$ is actually archimedean and, further, $K = \mathrm{coz}^{-1}[J]$, but these are easy consequences of the following result which is clearly of independent interest as well.

**Lemma 4.** *For any archimedean kernel $K$ of a function ring $\mathcal{R}L$, $\alpha \in K$ whenever $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ for some $\beta \in K$.*

**Proof.** Consider any such $\alpha, \beta \in \mathcal{R}L$, assuming $\alpha, \beta \geqslant \mathbf{0}$ without loss of generality. Now, for any natural $n$,

$$\mathrm{coz}\bigl(\beta \vee (\mathbf{1} - n\alpha)^+\bigr) = \mathrm{coz}\bigl(n\alpha \vee (\mathbf{1} - n\alpha)^+\bigr) = \mathrm{coz}\bigl(n\alpha + (\mathbf{1} - n\alpha)^+\bigr) = \mathrm{coz}(\mathbf{1} \vee n\alpha) = e,$$

the first step since $\mathrm{coz}(\beta) = \mathrm{coz}(\alpha) = \mathrm{coz}(n\alpha)$. It follows that $\beta \vee (\mathbf{1} - n\alpha)^+$ is invertible so that

$$K \vee \bigl\langle(\mathbf{1} - n\alpha)^+\bigr\rangle = \langle\mathbf{1}\rangle$$

in $\mathfrak{K}\mathcal{R}L$ ($\langle\cdot\rangle$ for principal archimedean kernel) and hence $\langle(n\alpha - \mathbf{1})^+\rangle \subseteq K$ since $\langle\gamma^+\rangle \cap \langle\gamma^-\rangle = \langle\gamma^+ \wedge \gamma^-\rangle = \langle\mathbf{0}\rangle$ for any $\gamma \in \mathcal{R}L$. This shows $\alpha \in K$. □

## 4. Archimedean homomorphisms of Coz *L*

Here we single out specific homomorphisms of the lattice $\mathrm{Coz}\,L$, somewhat analogous to the notion of archimedean ideal.

For any completely regular frame $L$, an *archimedean homomorphism* of $\mathrm{Coz}\,L$ is a bounded lattice homomorphism $\tau : \mathrm{Coz}\,L \to M$ to any frame $M$ such that

$$\tau\bigl(\mathrm{coz}(\alpha)\bigr) = \bigvee\bigl\{\tau\bigl(\mathrm{coz}\bigl((n\alpha - \beta)^+\bigr)\bigr) \bigm| n = 1, 2, \ldots\bigr\}$$

for any $\alpha, \beta \geqslant \mathbf{0}$ in $\mathcal{R}L$.

Note that, by Lemma 1, the identical embedding $\mathrm{Coz}\,L \to L$ is of this kind. On the other hand, if $\tau : \mathrm{Coz}\,L \to M$ is an archimedean homomorphism, and $h : M \to N$ any frame homomorphism then the composite $h\tau : \mathrm{Coz}\,L \to N$ is an archimedean homomorphism. In particular, then, the restriction to $\mathrm{Coz}\,L$ of any frame homomorphism $L \to M$ is of this kind.

An archimedean homomorphism $\rho : \mathrm{Coz}\,L \to N$ will be called *universal* if any archimedean homomorphism $\tau : \mathrm{Coz}\,L \to M$ determines a unique frame homomorphism $h : N \to M$ such that $\tau = h\rho$. Needless to say, if such $\rho$ exists it is unique up to a unique isomorphism.

Regarding existence, we now have

**Lemma 5.** *For any completely regular frame $L$, the map $\mathrm{Coz}\,L \to \mathfrak{A}\,\mathrm{Coz}\,L$ taking $\mathrm{coz}(\alpha)$ to $\downarrow\mathrm{coz}(\alpha)$ is the universal archimedean homomorphism of $\mathrm{Coz}\,L$.*

**Proof.** Given that the principal ideals in $\mathrm{Coz}\,L$ are archimedean,

$$\downarrow\mathrm{coz}(\alpha) = \bigvee\bigl\{\downarrow\mathrm{coz}\bigl((n\alpha - \beta)^+\bigr) \bigm| n = 1, 2, \ldots\bigr\}$$

in $\mathfrak{A}\,\mathrm{Coz}\,L$ for any $\alpha, \beta \geqslant \mathbf{0}$ in $\mathcal{R}L$, and since $\mathrm{coz}(\alpha) \mapsto \downarrow\mathrm{coz}(\alpha)$ is also a bounded lattice homomorphism it is an archimedean homomorphism $\mathrm{Coz}\,L \to \mathfrak{A}\,\mathrm{Coz}\,L$. Further, any such $\tau : \mathrm{Coz}\,L \to M$, as a bounded lattice homomorphism, determines the frame homomorphism $\bar{\tau} : \mathfrak{J}\,\mathrm{Coz}\,L \to M$ such that $\bar{\tau}(J) = \bigvee \tau[J]$, and then $\bar{\tau}(k_0(J)) = \bar{\tau}(J)$ for the prenucleus $k_0$ introduced in the proof of Lemma 2: if $\mathrm{coz}((n\alpha - \beta)^+) \in J$ for some $\alpha, \beta \geqslant \mathbf{0}$ and all $n$ then $\tau(\mathrm{coz}(\alpha)) \leqslant \bar{\tau}(J)$ by the definition of $\tau$. It follows that $\bar{\tau}k = \bar{\tau}$ for the nucleus $k$ corresponding to $\mathfrak{A}\,\mathrm{Coz}\,L$ and hence the restriction of $\bar{\tau}$ to $\mathfrak{A}\,\mathrm{Coz}\,L$ is a frame homomorphism $\tilde{\tau} : \mathfrak{A}\,\mathrm{Coz}\,L \to M$, obviously such that

$$\tilde{\tau}\bigl(\downarrow\mathrm{coz}(\alpha)\bigr) = \bar{\tau}\bigl(\downarrow\mathrm{coz}(\alpha)\bigr) = \tau\bigl(\mathrm{coz}(\alpha)\bigr).$$

Finally, this $\tilde{\tau}$ is unique since the $\downarrow\mathrm{coz}(\alpha)$ generate $\mathfrak{A}\,\mathrm{Coz}\,L$. □

$\mathcal{R}$-completeness can now be characterized in terms of archimedean homomorphisms as follows.

**Proposition 2.** *A completely regular frame $L$ is $\mathcal{R}$-complete iff any archimedean homomorphism $\tau : \operatorname{Coz} L \to M$ extends to a frame homomorphism $h : L \to M$.*

**Proof.** ($\Rightarrow$) For $\mathcal{R}$-complete $L$, $j_L : \mathfrak{A}\operatorname{Coz} L \to L$ is an isomorphism so that we have the frame homomorphism

$$h = \tilde{\tau}(j_L)^{-1} : L \to M$$

with $\tilde{\tau} : \mathfrak{A}\operatorname{Coz} L \to M$ as in the proof of Lemma 5, and then

$$h\big(\operatorname{coz}(\alpha)\big) = \tilde{\tau}\big(\downarrow\operatorname{coz}(\alpha)\big) = \tau\big(\operatorname{coz}(\alpha)\big),$$

showing $h$ extends $\tau$.

($\Leftarrow$) Since the map $\operatorname{coz}(\alpha) \mapsto \downarrow\operatorname{coz}(\alpha)$ is an archimedean homomorphism $\operatorname{Coz} L \to \mathfrak{A}\operatorname{Coz} L$ it extends to a frame homomorphism $h : L \to \mathfrak{A}\operatorname{Coz} L$. Further $hj_L = id_{\mathfrak{A}\operatorname{Coz} L}$ because

$$hj_L\big(\downarrow\operatorname{coz}(\alpha)\big) = h\big(\operatorname{coz}(\alpha)\big) = \ \downarrow\operatorname{coz}(\alpha),$$

and since $j_L$ is onto this shows it is an isomorphism, making $L$ $\mathcal{R}$-complete. □

Next, recall that a *realcompact frame* is a completely regular frame $L$ for which any ring homomorphism $\varphi : \mathcal{R}L \to \mathbf{R}$ is $\mathcal{R}\xi$ for some homomorphism $\xi : L \to \mathbf{2}$ (where use is made of the fact that $\mathbf{R} = \mathcal{R}\mathbf{2}$). Further, Proposition 2 together with various connected results (Banaschewski [2, Appendix 3]; Banaschewski [4, Proposition 3]) then leads to the following characterization:

*$L$ is realcompact iff every homomorphism $\xi : \mathfrak{A}\operatorname{Coz} L \to \mathbf{2}$ factors through $j_L : \mathfrak{A}\operatorname{Coz} L \to L$.*

Based on this, we now obtain

**Proposition 3.** *A completely regular frame $L$ is realcompact iff any archimedean homomorphism $\operatorname{Coz} L \to \mathbf{2}$ extends to a frame homomorphism $L \to \mathbf{2}$.*

**Proof.** ($\Rightarrow$) Any archimedean homomorphism $\tau : \operatorname{Coz} L \to \mathbf{2}$ determines a frame homomorphism $\xi : \mathfrak{A}\operatorname{Coz} L \to \mathbf{2}$ such that $\xi(\downarrow\operatorname{coz}(\alpha)) = \tau(\operatorname{coz}(\alpha))$ by Lemma 5. Further, realcompactness then supplies a frame homomorphism $\zeta : L \to \mathbf{2}$ such that $\xi = \zeta j_L$, and this, in turn, extends $\tau$:

$$\zeta\big(\operatorname{coz}(\alpha)\big) = \zeta j_L\big(\downarrow\operatorname{coz}(\alpha)\big) = \xi\big(\downarrow\operatorname{coz}(\alpha)\big) = \tau\big(\operatorname{coz}(\alpha)\big).$$

($\Leftarrow$) For any homomorphism $\xi : \mathfrak{A}\operatorname{Coz} L \to \mathbf{2}$, $\operatorname{coz}(\alpha) \mapsto \xi(\downarrow\operatorname{coz}(\alpha))$ is an archimedean homomorphism $\operatorname{Coz} L \to \mathbf{2}$ by Lemma 5, and if $\zeta : L \to \mathbf{2}$ is its extension to a frame homomorphism given by the present hypothesis then

$$\zeta j_L\big(\downarrow\operatorname{coz}(\alpha)\big) = \zeta\big(\operatorname{coz}(\alpha)\big) = \xi\big(\downarrow\operatorname{coz}(\alpha)\big)$$

which shows $\zeta j_L = \xi$ and hence the realcompactness of $L$. □

To interpret the above result in the context of classical topology, recall first that, for any space $X$ and its frame $\mathfrak{O}X$ of open sets, the usual function ring $C(X)$ is isomorphic to $\mathcal{R}(\mathfrak{O}X)$ such that the cozero set lattice $\operatorname{Coz} X$ of $X$ is isomorphic to the present $\operatorname{Coz}(\mathfrak{O}X)$: $a \in C(X)$ corresponds to $\alpha \in \mathcal{R}(\mathfrak{O}X)$ such that

$$\alpha(p,q) = a^{-1}\big[\{\lambda \in \mathbb{R} \mid p < \lambda < q\}\big]$$

and then obviously

$$\operatorname{coz}(\alpha) = a^{-1}\big[\{\lambda \in \mathbb{R} \mid \lambda \neq 0\}\big].$$

Further, for any point $x \in X$, the map $\hat{x} : \operatorname{Coz} X \to \mathbf{2}$ such that $\hat{x}(U) = 1$ iff $x \in U$ is an archimedean homomorphism, as the restriction to $\operatorname{Coz} X$ of the corresponding map $\mathfrak{O}X \to \mathbf{2}$ which is a frame homomorphism. We call these archimedean homomorphism $\hat{x}$ *fixed*.

Now, given that (i) a Tychonoff space $X$ is realcompact in the classical sense iff $\mathfrak{O}X$ is realcompact in the present sense and (ii) any frame homomorphism $\mathfrak{O}X \to \mathbf{2}$ is given by some $x \in X$ since $X$ is Hausdorff, we have the following immediate consequence of Proposition 3:

**Corollary 1.** *A Tychonoff space $X$ is realcompact iff every archimedean homomorphism $\operatorname{Coz} X \to \mathbf{2}$ is fixed.*

## 5. The integer-valued case

It obviously makes sense to consider the analogues of the preceding notions for the functor $\mathcal{Z}$ on the category **ODFrm** of zero-dimensional frames which takes each $L$ to the $\ell$-ring $\mathcal{Z}L$ of *the integer-valued continuous functions* on $L$, but it turns out that much of this has already been done (Banaschewski [3]) so that a brief account will be sufficient to describe the situation. Regarding this, it should be noted that [3] treats the subject in a slightly different form, specifically without explicit reference to the functor $\mathcal{Z}$, but it is clear that what is done there may be described equivalently in the style adopted below.

The $\mathcal{Z}L$, it will be recalled, have as their elements the frame homomorphisms $\mathfrak{P}\mathbf{Z} \to L$, conveniently described as the maps $\alpha : \mathbf{Z} \to L$ such that

$$\alpha(k) \wedge \alpha(\ell) = 0 \quad \text{if } k \neq \ell \quad \text{and} \quad \bigvee\{\alpha(m) \mid m \in \mathbf{Z}\} = e,$$

while their operations are derived from the $\ell$-ring operations of $\mathbf{Z}$ such that

$$\text{for } \diamond = +, \cdot, \wedge, \vee\colon\ \alpha \diamond \beta(m) = \bigvee\{\alpha(k) \wedge \beta(\ell) \mid k \diamond \ell = m\},$$

$$(-\alpha)(m) = \alpha(-m),$$

$$\mathbf{k}(m) = e \quad \text{if } m = k \text{ (and hence 0 otherwise) for } k = 0, 1.$$

It is evident from this definition that these operations satisfy all identities which hold for their counterparts in $\mathbf{Z}$. In particular, they make the $\mathcal{Z}L$ commutative $f$-rings with unit such that $\alpha \wedge (\mathbf{1} - \alpha) \leqslant \mathbf{0}$ for all $\alpha \in \mathcal{Z}L$, called $\mathbf{Z}$-rings; moreover, they are archimedean.

Clearly, then, we have the corresponding notions of $\mathcal{Z}$-isomorphism, $\mathcal{Z}$-completeness, and $\mathcal{Z}$-completion, as well as the present version of the cozero map, $\operatorname{coz} : \mathcal{Z}L \to L$ such that

$$\operatorname{coz}(\alpha) = \bigvee\{\alpha(m) \mid 0 \neq m \in \mathbf{Z}\}$$

where the latter has the same properties as the earlier map $\mathcal{R}L \to L$.

Note that, specific to the present setting, any $\operatorname{coz}(\alpha)$ is complemented (with complement $\alpha(0)$), and conversely because any complemented $c \in L$ determines its *characteristic function* $\gamma \in \mathcal{Z}L$ such that

$$\gamma(1) = c, \qquad \gamma(0) = c^*, \qquad \gamma(m) = 0 \text{ otherwise}$$

and hence $c = \operatorname{coz}(\gamma)$. As a result, what corresponds to the lattice $\operatorname{Coz} L$ in the present situation, is just the *Boolean part* $BL$ of $L$, that is, the Boolean algebra of complemented elements of $L$. Further, the ideals which take over the rôle of the archimedean ideals of $\operatorname{Coz} L$ are the ideals $J$ of $BL$ such that

*$a \in BL$ and $a = \bigvee S$ in $L$ for some countable $S \subseteq J$ implies $a \in J$.*

Finally, these $J$ form a zero-dimensional frame $\mathfrak{H}BL$, evidently generated by the principal ideals of $BL$, such that taking joins in $L$ determines a homomorphism $h_L : \mathfrak{H}BL \to L$ which is the coreflection map to $L$ from $\mathcal{Z}$-complete frames (Banaschewski [3]). That it is also the unique $\mathcal{Z}$-isomorphism to $L$ with $\mathcal{Z}$-complete domain was not included in this earlier work but can easily be deduced.

Regarding the counterpart of Section 4, some of this is in fact new. In the present setting, what corresponds to the archimedean homomorphisms of the $\operatorname{Coz} L$ are the bounded lattice homomorphisms $\tau : BL \to M$, $M$ any frame, such that $\tau(a) = \bigvee \tau[S]$ whenever $a \in BL$ and $a = \bigvee S$ in $L$ for some countable $S \subseteq BL$, called the *$\sigma$-homomorphisms of $BL$* (Banaschewski–Gilmour [5]). Note that the restriction to $BL$ of any frame homomorphism $L \to M$ is evidently of this kind, as is any composite of such a map with a frame homomorphism.

Now we have the following analogue of Lemma 5, with the obvious notion of universality.

**Lemma 6.** *For any zero-dimensional frame $L$, the map $BL \to \mathfrak{H}BL$ taking $c \in BL$ to its principal ideal $\downarrow c$ is the universal $\sigma$-homomorphism of $BL$.*

**Proof.** Any $\tau : BL \to M$ of the type in question determines a frame homomorphism $\bar{\tau} : \mathfrak{J}BL \to M$ such that $\bar{\tau}(J) = \bigvee \tau[J]$. Further $\mathfrak{H}BL \subseteq \mathfrak{J}BL$ is the closure system of the ideals fixed by the prenucleus $n_0$ such that

$$n_0(J) = \left\{a \in BL \;\middle|\; a = \bigvee S \text{ in } L \text{ for some countable } S \subseteq J\right\}$$

where it is clear that $\tau(a) \leqslant \bigvee \tau[J]$ for all $a$ as indicated so that $\bar{\tau}(n_0(J)) = \bar{\tau}(J)$. Consequently, the restriction of $\bar{\tau}$ to $\mathfrak{H}BL$ is a frame homomorphism $\tilde{\tau} : \mathfrak{H}BL \to M$ such that

$$\tilde{\tau}(\downarrow c) = \bar{\tau}(\downarrow c) = \tau(c),$$

unique because the $\downarrow c$ generate $\mathfrak{H}BL$. $\square$

It is now immediate that formally the same proof as the earlier one of Proposition 3 leads to

**Proposition 4.** *A zero-dimensional frame $L$ is $\mathcal{Z}$-complete iff every $\sigma$-homomorphism $BL \to M$ extends to a frame homomorphism $L \to M$.*

Similarly, recalling that a **Z**-*compact* frame is a zero-dimensional frame $L$ for which any ring homomorphism $\varphi : \mathcal{Z}L \to \mathbf{Z}$ is $\mathcal{Z}\xi$ for some $\xi : L \to \mathbf{2}$ and that these $L$ are characterized by the condition that any homomorphism $\mathfrak{H}\operatorname{Coz} L \to \mathbf{2}$ factor through $h_L : \mathfrak{H}BL \to L$, we further obtain the result given as Proposition 11 in Banaschewski–Gilmour [5]:

*A zero-dimensional frame is **Z**-compact iff every $\sigma$-homomorphism $BL \to \mathbf{2}$ extends to a frame homomorphism $L \to \mathbf{2}$.*

## 6. The effect of the Axiom of Countable Choice

*Throughout this section, the axiom in question will be assumed.* The main purpose here is to demonstrate that, under this assumption, the results presented earlier immediately lead to the familiar facts previously known in this area.

For this, it will be useful to recall how ACC is usually employed to obtain that $\operatorname{Coz} L$ is a sub-$\sigma$-frame of $L$, for any frame $L$.

Given any countable family $(c_n)_{n\in\omega}$ in $\operatorname{Coz} L$, one takes $\alpha_n \in \mathcal{R}L$ such that $\mathbf{0} \leqslant \alpha_n \leqslant \mathbf{1}$ and $\operatorname{coz}(\alpha_n) = c_n$ and then considers $\alpha = \sum_0^\infty \frac{\alpha_n}{2^n}$, using the fact that the bounded part of $\mathcal{R}L$ ($\gamma \in \mathcal{R}L$ such that $|\gamma| \leqslant \mathbf{n}$ for some $\mathbf{n}$) is complete in the usual uniform topology of bounded $f$-rings. Then one has, for any $n$,

$$\alpha \leqslant \sum_0^n \frac{\alpha_k}{2^k} + \sum_{k\geqslant n+1} \frac{1}{2^k} = \sum_0^n \frac{\alpha_k}{2^k} + \frac{1}{2^n},$$

hence

$$\left(\alpha - \frac{1}{2^n}\right)^+ \leqslant \sum_0^n \frac{\alpha_k}{2^k}$$

so that

$$\operatorname{coz}\big((n\alpha - \mathbf{1})^+\big) \leqslant \operatorname{coz}\big(\big(2^n\alpha - \mathbf{1}\big)^+\big) = \operatorname{coz}\left(\left(\alpha - \frac{1}{2^n}\right)^+\right) \leqslant \operatorname{coz}\left(\sum_0^n \frac{\alpha_k}{2^k}\right) = \bigvee_0^n \operatorname{coz}(\alpha_k) = \bigvee_0^n c_k \tag{$*$}$$

and finally by Lemma 1

$$\operatorname{coz}(\alpha) = \bigvee_0^\infty \operatorname{coz}\big((n\alpha - \mathbf{1})^+\big) \leqslant \bigvee_0^\infty c_n.$$

On the other hand,

$$\bigvee_0^n c_k = \operatorname{coz}\left(\sum_0^n \frac{\alpha_k}{2^k}\right) \leqslant \operatorname{coz}(\alpha)$$

since $\sum_0^n \frac{\alpha_k}{2^k} \leqslant \alpha$, showing in all that $\operatorname{coz}(\alpha) = \bigvee_0^\infty c_n$.

The crucial points are now given by

**Proposition 5.** *For any completely regular frame $L$,*

(1) *any archimedean ideal $J$ of* $\operatorname{Coz} L$ *is a $\sigma$-ideal, and*
(2) *any archimedean homomorphism $\tau : \operatorname{Coz} L \to M$ is a $\sigma$-frame homomorphism.*

**Proof.** (1) For any family $(c_n)_{n\in\omega}$ in $J$, if $\alpha \in \mathcal{R}L$ is as described above then $\operatorname{coz}((\mathbf{n}\alpha - \mathbf{1})^+) \in J$ for any $n$ by $(*)$ and hence $\bigvee_0^\infty c_n = \operatorname{coz}(\alpha) \in J$.

(2) Again, given any family $(c_n)_{n\in\omega}$ in $\operatorname{Coz} L$, take $\alpha \in \mathcal{R}L$ as before. Then

$$\tau\left(\bigvee_0^\infty c_n\right) = \tau\big(\operatorname{coz}(\alpha)\big) = \bigvee\big\{\tau\big(\operatorname{coz}\big((n\alpha - \mathbf{1})^+\big)\big) \bigm| n = 1, 2, \ldots, \big\} \leqslant \bigvee_0^\infty \tau(c_n),$$

the last step again by $(*)$, showing that $\tau$ preserves the join of any countable subset of $\operatorname{Coz} L$. $\square$

Now, given that any $\sigma$-ideal and any $\sigma$-frame homomorphism of $\operatorname{Coz} L$ are trivially archimedean, we have the following immediate consequences of Propositions 1–3:

**Corollary 2.** *For any completely regular frame $L$,*

(1) *the homomorphism $\mathfrak{H}\operatorname{Coz} L \to L$ given by taking joins in $L$ is the coreflection map to $L$ from $\mathcal{R}$-complete frames,*
(2) *$L$ is $\mathcal{R}$-complete iff every $\sigma$-frame homomorphism $\operatorname{Coz} L \to M$ extends to a frame homomorphism $L \to M$.*
(3) *$L$ is realcompact iff every $\sigma$-frame homomorphism $\operatorname{Coz} L \to \mathbf{2}$ extends to a frame homomorphism.*

*Further, a Tychonoff space $X$ is realcompact iff every $\sigma$-frame homomorphism $\operatorname{Coz} X \to \mathbf{2}$ is fixed.*

Taking into account that $\mathcal{R}$-complete means Lindelöf in the present context, (1) is due to Madden–Vermeer [8] and (2) to Banaschewski–Gilmour [5]. Further, [5] also contains (3), albeit involving a formally different but equivalent notion of realcompactness. Finally, the last result is obviously related to that of Hewitt [6] that a Tychonoff space is realcompact iff any $\sigma$-frame homomorphism from its $\sigma$-field of Baire sets into $\mathbf{2}$ is fixed.

**Remark 2.** Somewhat unrelated to the present topic, we note that the calculations at the beginning of this section can also be used to show that an ideal $J$ in $\operatorname{Coz} L$ is already archimedean if it only satisfies the weaker condition that $\operatorname{coz}(\alpha) \in J$ for $\alpha \geqslant \mathbf{0}$ whenever $\operatorname{coz}((n\alpha - \mathbf{1})^+) \in J$ for all $n$. To see this, consider any $\alpha, \beta \geqslant \mathbf{0}$ such that $\operatorname{coz}((n\alpha - \beta)^+) \in J$ for all $n$ and put

$$\gamma = \sum_0^\infty \frac{(n\alpha - \beta)^+ \wedge \mathbf{1}}{2^n}.$$

Then, as earlier,

$$\operatorname{coz}\big((n\gamma - \mathbf{1})^+\big) \leqslant \bigvee_0^n \operatorname{coz}\left(\frac{(k\alpha - \beta)^+ \wedge \mathbf{1}}{2^k}\right) = \bigvee_0^n \operatorname{coz}\big((k\alpha - \beta)^+\big) = \operatorname{coz}\big((n\alpha - \beta)^+\big),$$

showing $\operatorname{coz}((n\gamma - \mathbf{1})^+) \in J$ for all $n$ so that $\operatorname{coz}(\gamma) \in J$ by the present hypothesis. On the other hand, by our earlier result and Lemma 1,

$$\operatorname{coz}(\gamma) = \bigvee\left\{\operatorname{coz}\left(\frac{(n\alpha - \beta)^+ \wedge \mathbf{1}}{2^n}\right) \,\Big|\, n = 1, 2, \ldots\right\} = \bigvee\big\{\operatorname{coz}\big((n\alpha - \beta)^+\big) \,\big|\, n = 1, 2, \ldots\big\} = \operatorname{coz}(\alpha)$$

and hence $\operatorname{coz}(\alpha) \in J$, as claimed.

In closing we note that, in the case of the functor $\mathcal{Z}$ considered in Section 5, ACC has a substantially different impact. In particular, it does not affect the properties of $BL$, the description of the frame $\mathfrak{H}BL$, or the nature of the $\sigma$-homomorphisms $BL \to M$. It does, however, cause one change: as in the case of $\mathfrak{A}\operatorname{Coz} L$, it makes $\mathfrak{H}BL$ Lindelöf, and again that condition is equivalent to ACC (Banaschewski [3]).

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## References

[1] B. Banaschewski, Another look at the localic Tychonoff theorem, Comment. Math. Univ. Carolin. 29 (1988) 647–656.
[2] B. Banaschewski, The Real Numbers in Pointfree Topology, Textos Mat. Ser. B, vol. 12, Departamento de Matemática da Universidade de Coinbra, 1997.
[3] B. Banaschewski, The Axiom of Countable Choice and pointfree topology, Appl. Categ. Structures 9 (2001) 245–258.
[4] B. Banaschewski, On the function ring functor in pointfree topology, Appl. Categ. Structures 13 (2005) 305–328.
[5] B. Banaschewski, C. Gilmour, Realcompactness and the cozero part of a frame, Appl. Categ. Structures 9 (2001) 395–417.
[6] E. Hewitt, Linear functionals on spaces of continuous functions, Fund. Math. 37 (1950) 161–189.
[7] P.T. Johnstone, Stone Spaces, Cambridge Stud. Adv. Math., vol. 3, Cambridge University Press, 1982.
[8] J. Madden, J. Vermeer, Lindelöf locales and realcompactness, Math. Proc. Cambridge Philos. Soc. 99 (1986) 473–480.
[9] A. Pultr, Frames, Handbook of Algebra, vol. 3, Elsevier Science B.V., 2003.
[10] S. Vickers, Topology Via Logic, Cambridge Tracts Theoret. Comput. Sci., Cambridge University Press, 1985.